On general (α, β)-metrics with isotropic Berwald curvature

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Abstract

In this paper, we study a class of Finsler metrics called general (α, β)-metrics, which are defined by a Riemannian metric α and a 1-form β. We classify this class of Finsler metrics with isotropic Berwald curvature under certain condition.

1 Introduction

In Finsler geometry, the Berwald curvature is an important non-Riemannian quantity. A Finsler metric F on a manifold M is said to be of *isotropic Berwald curvature* if its Berwald curvature $B_j{}^i{}_{kl}$ satisfies

$$B_j{}^i{}_{kl} = \tau(x)(F_{y^j y^k}\delta^i{}_l + F_{y^j y^l}\delta^i{}_k + F_{y^l y^k}\delta^i{}_j + F_{y^j y^k y^l} y^i), \tag{1.1}$$

where $\tau(x)$ is a scalar function on M. A Finsler metric is called a *Berwald metric* if $\tau(x) = 0$. Berwald metrics are just a bit more general than Riemannian and locally Minkowskian metrics. A Berwald space is that all tangent spaces are linearly isometric to a common Minkowski space.

Chen-Shen showed that a Finsler metric F is of isotropic Berwald curvature if and only if it is a Douglas metric with isotropic mean Berwald curvature [3]. Tayebi-Rafie's result tells us that every isotropic Berwald metric is of isotropic S-curvature [8]. In [9], Tayebi-Najafi proved that isotropic Berwald metrics of scalar flag curvature are of Randers type. Recently, Guo-Liu-Mo have shown that every spherically symmetric Finsler metric of isotropic Berwald curvature is a Randers metric [4].

Randers metrics , which are introduced by a physicist G. Randers in 1941 when he studied general relativity, are an important class of Finsler metrics. It is of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form. Bao-Robles-Shen show that Randers metric $F = \alpha + \beta$, where

$$\alpha := \frac{\sqrt{\varepsilon^2(xu + yv + zw)^2) + (u^2 + v^2 + w^2)[1 - \varepsilon^2(x^2 + y^2 + z^2)]}}{1 - \varepsilon^2(x^2 + y^2 + z^2)},$$

$$\beta := \frac{-\varepsilon(xu + yv + zw)}{1 - \varepsilon^2(x^2 + y^2 + z^2)}.$$

is of constant S-curvature and satisfies $d\beta = 0$. Hence, F is of isotropic Berwald curvature[1, 4].

Many famous Finsler metrics can be expressed in the following form

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big), \tag{1.2}$$

where α is a Riemannian metric, β is a 1-form, $b := \|\beta_x\|_\alpha$ and $\phi(b^2, s)$ is a smooth function. Finsler metrics in this form are called general (α, β)-metrics [10, 11]. If $\phi = \phi(s)$ is independent of b^2, then $F = \alpha\phi(\frac{\beta}{\alpha})$ is an (α, β)-metric. If $\alpha = |y|$, $\beta = \langle x, y\rangle$, then $F = |y|\phi(|x|^2, \frac{\langle x,y\rangle}{|y|})$ is the so-called spherically symmetric Finsler

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metrics [6, 12]. Moreover, general (α,β)-metrics include part of Bryant's metrics [2, 10] and part of generalized fourth root metrics [5]. In fact, Randers metrics $F=\alpha+\beta$ can also be expressed in the following navigation form

$$F=\frac{\sqrt{(1-\bar{b}^2)\bar{\alpha}^2+\bar{\beta}^2}}{1-\bar{b}^2}+\frac{\bar{\beta}}{1-\bar{b}^2},$$

where $\bar{\alpha}$ is also a Riemannian metric, $\bar{\beta}$ is a 1-form and $\bar{b}:=\|\bar{\beta}_x\|_{\bar{\alpha}}$. $(\bar{\alpha},\bar{\beta})$ is called *the navigation data* of the Randers metric F. In [8], Tayebi-Rafie showed that if a Randers metric $F=\alpha+\beta$ is an non-trivial isotropic Berwald metric, then $\bar{\beta}$ is a conformal 1-form with respect to $\bar{\alpha}$, namely, $\bar{\beta}$ satisfies

$$\bar{b}_{i|j}+\bar{b}_{j|i}=c(x)\bar{a}_{ij},$$

where $c(x)\neq 0$ and $\bar{b}_{i|j}$ is the covariant derivation of $\bar{\beta}$ with respect to $\bar{\alpha}$.

In this paper, we mainly study general (α,β)-metrics $F=\alpha\phi(b^2,s)$ with isotropic Berwald curvature and show the following classification theorem

Theorem 1.1. *Let $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β satisfies*

$$b_{i|j}=ca_{ij}, \tag{1.3}$$

where $c=c(x)\neq 0$ is a scalar function on M . If F is of isotropic Berwald curvature, then one of the following holds

(1) F is Riemannian;

(2) F is a Randers metric;

(3) F is a Kropina metric;

(3) F is a Berwald metric which can be expressed by

$$F=\alpha\varphi\Big(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2}\Big)e^{\int(\frac{1}{2}b^2t_2-\frac{1}{b^2})db^2}s.$$

where $\varphi(\cdot)$ is any positive continuously differentiable function and t_2 is a smooth function of b^2.

Remark: 1) we assume that β is closed and conformal with respect to α, i.e. (1.3) holds. According to the relate discussions for isotropic Berwald metrics [3, 4, 8, 9], we believe that the assumption here is reasonable and appropriate.

2) It should be pointed out that if the scalar function $c(x)=0$, then according to Proposition 3.1, $B_j{}^i{}_{kl}=0$, namely, $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ is a Berwald metric for any function $\phi(b^2,s)$. So it will be regarded as a trivial case.

2 Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, which are defined by

$$G^i=\frac{1}{4}g^{il}\left\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\right\},$$

where $(g^{ij}):=\left(\frac{1}{2}[F^2]_{y^iy^j}\right)^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$.

Lemma 2.1. *[10] Let $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ be a general (α,β)-metric on an n-dimensional manifold M. Then the function F is a regular Finsler metric for any Riemannian metric α and any 1-form β if and only if $\phi(b^2,s)$ is a positive smooth function defined on the domain $|s|\leq b<b_o$ for some positive number (maybe infinity) b_o satisfying*

$$\phi-s\phi_2>0,\quad \phi-s\phi_2+(b^2-s^2)\phi_{22}>0, \tag{2.1}$$

when $n \geq 3$ *or*

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.2}$$

when $n = 2$.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Denote the coefficients of the covariant derivative of β with respect to α by $b_{i|j}$, and let

$$r_{ij} = \tfrac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \tfrac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^i y^j,\ s^i{}_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^j r_{ji},\ s_i = b^j s_{ji},\ r_0 = r_i y^i,\ s_0 = s_i y^i,\ r^i = a^{ij}r_j,\ s^i = a^{ij}s_j,\ r = b^i r_i,$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. It is easy to see that β is closed if and only if $s_{ij} = 0$.

Lemma 2.2. *[10] the spray coefficients G^i of a general (α, β)-metric $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ are related to the spray coefficients ${}^\alpha G^i$ of α and given by*

$$\begin{aligned} G^i \ =\ & {}^\alpha G^i + \alpha Q s^i{}_0 + \big\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\big\}\frac{y^i}{\alpha} \\ & + \big\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\big\}\, b^i - \alpha^2 R(r^i + s^i), \end{aligned} \tag{2.3}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2},$$
$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Psi = \frac{\phi_{22}}{2\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)},$$
$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2. In the following, we will introduce an important non-Riemannian quantity.

Definition 2.3. [7] Let

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \tag{2.4}$$

where G^i are the spray coefficients of F. The tensor $B := B_j{}^i{}_{kl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called *Berwald tensor.* A Finsler metric is called *a Berwald metric* if the Berwald tensor vanishes, i.e. the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_xM$ at every point $x \in M$.

3 Berwald curvature of general (α, β)-metrics

In this section, we will compute the Berwald curvature of a general (α, β)-metric.

Proposition 3.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies (1.3), then the Berwald curvature of F is given by*

$$\begin{aligned} B_j{}^i{}_{kl} \ =\ & \frac{c}{\alpha}\left\{\left[(E - sE_2)a_{kl} + E_{22}b_l b_k\right]\delta^i{}_j + \frac{1}{\alpha^2}\left[\frac{s}{\alpha}(3E_{22} + sE_{222})y_l y_j - (E_{22} + sE_{222})b_l y_j\right] b_k y^i\right\}(k \to l \to j \to k) \\ & -\frac{c}{\alpha^2}\left\{sE_{22}\left[(y_k b_l + y_l b_k)\delta^i{}_j + a_{jl}b_k y^i\right] + \frac{1}{\alpha}(E - sE_2 - s^2E_{22})(y_l\delta^i{}_j + a_{lj}y^i)y_k\right\}(k \to l \to j \to k) \\ & +\frac{c}{\alpha^2}\left[\frac{1}{\alpha^3}(3E - 3sE_2 - 6s^2E_{22} - s^3E_{222})y_k y_j y_l + E_{222}b_l b_k b_j\right] y^i \\ & +\frac{c}{\alpha}\left[(H_2 - sH_{22})(b_j - \frac{s}{\alpha}y_j)a_{kl} - \frac{1}{\alpha^2}(H_2 - sH_{22} - s^2H_{222})b_l y_j y_k - \frac{sH_{222}}{\alpha}b_k b_l y_j\right] b^i (k \to l \to j \to k) \\ & +\frac{c}{\alpha}\left[\frac{s}{\alpha^3}(3H_2 - 3sH_{22} - s^2H_{222})y_j y_k y_l + H_{222}b_l b_k b_j\right] b^i, \end{aligned} \tag{3.1}$$

where $y_i := a_{ij}y^j$ *and* $b^i := a^{ij}b_j$, $c = c(x) \neq 0$ *is a scalar function on* M.

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - H\frac{s\phi + (b^2 - s^2)\phi_2}{\phi}, \tag{3.2}$$

$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\big[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big]}. \tag{3.3}$$

Proof. By (1.3), we have

$$r_{00} = c\alpha^2, r_0 = c\beta, r = cb^2, r^i = cb^i, s^i{}_0 = 0, s_0 = 0, s^i = 0. \tag{3.4}$$

Substituting (3.4) into (2.3) yields

$$\begin{aligned} G^i &= {}^\alpha G^i + c\alpha\left\{\Theta(1 + 2Rb^2) + s\Omega\right\}y^i + c\alpha^2\left\{\Psi(1 + 2Rb^2) + s\Pi - R\right\}b^i \\ &= {}^\alpha G^i + c\alpha E y^i + c\alpha^2 H b^i, \end{aligned} \tag{3.5}$$

where E and H are given by (3.2) and (3.3) respectively. Note that

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \quad s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2}, \tag{3.6}$$

where $y_i := a_{ij}y^j$. Put

$$W^i := \alpha E y^i + \alpha^2 H b^i. \tag{3.7}$$

Differentiating (3.7) with respect to y^j yields

$$\frac{\partial W^i}{\partial y^j} = \alpha E \delta^i{}_j + (E\alpha_{y^j} + \alpha E_2 s_{y^j})y^i + \left\{[\alpha^2]_{y^j}E + \alpha^2 E_2 s_{y^j}\right\}b^i. \tag{3.8}$$

Differentiating (3.8) with respect to y^k yields

$$\begin{aligned} \frac{\partial^2 W^i}{\partial y^j \partial y^k} &= \big[(E\alpha_{y^k} + \alpha E_2 s_{y^k})\delta^i{}_j + E_2 s_{y^k}\alpha_{y^j}y^i + H_2[\alpha^2]_{y^j}s_{y^k}b^i\big](k \leftrightarrow j) \\ &\quad + \big(E\alpha_{y^j y^k} + \alpha E_{22}s_{y^k}s_{y^j} + \alpha E_2 s_{y^j y^k}\big)y^i \\ &\quad + \big\{[\alpha^2]_{y^j y^k}H + \alpha^2 H_{22}s_{y^k}s_{y^j} + \alpha^2 H_2 s_{y^j y^k}\big\}b^i, \end{aligned} \tag{3.9}$$

where $k \leftrightarrow j$ denotes symmetrization. Therefore, it follows from (3.9) that

$$\begin{aligned} \frac{\partial^3 W^i}{\partial y^j \partial y^k \partial y^l} &= \big[E_2(\alpha_{y^k}s_{y^l} + \alpha_{y^l}s_{y^k} + \alpha s_{y^k y^l}) + E\alpha_{y^k y^l} + \alpha E_{22}s_{y^l}s_{y^k}\big]\delta^i{}_j(k \to l \to j \to k) \\ &\quad + \big[E_2(s_{y^k}\alpha_{y^j y^l} + \alpha_{y^k}s_{y^j y^l}) + E_{22}(\alpha_{y^k}s_{y^j} + \alpha s_{y^k y^j})s_{y^l}\big]y^i(k \to l \to j \to k) \\ &\quad + \big\{H_2\big([\alpha^2]_{y^k y^l}s_{y^j} + [\alpha^2]_{y^k}s_{y^j y^l}\big) + H_{22}\big([\alpha^2]_{y^k}s_{y^l}s_{y^j} + \alpha^2 s_{y^k y^l}s_{y^j}\big)\big\}b^i(k \to l \to j \to k) \\ &\quad + \big(E\alpha_{y^j y^k y^l} + \alpha E_{222}s_{y^j}s_{y^k}s_{y^l} + \alpha E_2 s_{y^j y^k y^l}\big)y^i \\ &\quad + \big\{H[\alpha^2]_{y^j y^k y^l} + \alpha^2 H_{222}s_{y^j}s_{y^k}s_{y^l} + \alpha^2 H_2 s_{y^j y^k y^l}\big\}b^i, \end{aligned} \tag{3.10}$$

where $k \to l \to j \to k$ denotes cyclic permutation. It follows from (3.6) that

$$[\alpha^2]_{y^l} = 2y_l, \quad [\alpha^2]_{y^l y^j} = 2a_{lj}, \quad [\alpha^2]_{y^l y^j y^k} = 0, \tag{3.11}$$

$$\alpha_{y^l y^j} = \frac{1}{\alpha}\big(a_{lj} - \frac{y_l}{\alpha}\frac{y_j}{\alpha}\big), \quad \alpha_{y^l y^j y^k} = -\frac{1}{\alpha^3}[a_{kl}y_j(k \to l \to j \to k) - \frac{3}{\alpha^2}y_l y_j y_k], \tag{3.12}$$

$$s_{y^l y^j} = -\frac{1}{\alpha^2}\big[s a_{lj} + \frac{1}{\alpha}(b_l y_j + b_j y_l) - \frac{3s}{\alpha^2}y_l y_j\big], \tag{3.13}$$

$$s_{y^l y^j y^k} = \frac{1}{\alpha^5}\big\{[\alpha(3sy_j - \alpha b_j)a_{lk} + 3b_k y_l y_j](k \to l \to j \to k) - \frac{15s}{\alpha}y_k y_l y_j\big\}. \tag{3.14}$$

Plugging (3.11)-(3.14) into (3.10) yields

$$
\begin{aligned}
\frac{\partial^3 W^i}{\partial y^j \partial y^k \partial y^l} &= \frac{1}{\alpha}\left\{\big[(E-sE_2)a_{kl}+E_{22}b_l b_k\big]\delta^i{}_j+\frac{1}{\alpha^2}\big[\frac{s}{\alpha}(3E_{22}+sE_{222})y_l-(E_{22}+sE_{222})b_l\big]y_j b_k y^i\right\}(k\to l\to j\to k)\\
&-\frac{1}{\alpha^2}\left\{sE_{22}\big[(y_k b_l+y_l b_k)\delta^i{}_j+a_{jl}b_k y^i\big]+\frac{1}{\alpha}(E-sE_2-s^2E_{22})(y_l\delta^i_j+a_{jl}y^i)y_k\right\}(k\to l\to j\to k)\\
&+\frac{1}{\alpha^2}\left[\frac{1}{\alpha^3}(3E-3sE_2-6s^2E_{22}-s^3E_{222})y_k y_j y_l+E_{222}b_l b_k b_j\right]y^i\\
&+\frac{1}{\alpha}\left[(H_2-sH_{22})(b_j-\frac{s}{\alpha}y_j)a_{kl}-\frac{1}{\alpha^2}(H_2-sH_{22}-s^2H_{222})b_l y_j y_k-\frac{sH_{222}}{\alpha}b_k b_l y_j\right]b^i(k\to l\to j\to k)\\
&+\frac{1}{\alpha}\left[\frac{s}{\alpha^3}(3H_2-3sH_{22}-s^2H_{222})y_j y_k y_l+H_{222}b_l b_k b_j\right]b^i. \qquad (3.15)
\end{aligned}
$$

It follows from ${}^{\alpha}G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ that

$$
\frac{\partial^3 {}^{\alpha}G^i}{\partial y^j \partial y^k \partial y^l}=0. \qquad (3.16)
$$

By (2.4), (3.5), (3.7), (3.15) and (3.16), we obtain (3.1).

□

4 General (α,β)-metrics with isotropic Berwald curvature

In this section, we will classify general (α,β)-metrics with isotropic Berwald curvature under certain condition. Firstly, we show the following

Lemma 4.1. *Suppose that β satisfies (1.3), then a general (α,β)-metric $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ is of isotropic Berwald curvature if and only if*

$$
\begin{aligned}
E-sE_2-\rho(x)(\phi-s\phi_2)&=0, &(4.1)\\
H_2-sH_{22}&=0, &(4.2)
\end{aligned}
$$

where $\rho(x)=\frac{\tau(x)}{c(x)}$, E and H are given by (3.2) and (3.3) respectively. In particular, F is a Berwald metric if and only if $E-sE_2=H_2-sH_{22}=0$.

Proof. For a general (α,β)-metric $F=\alpha\phi(b^2,s)$, a direct computation yields

$$
\begin{aligned}
F_{y^j} &= \alpha_{y^j}\phi+\alpha\phi_2 s_{y^j},\\
F_{y^j y^k} &= \alpha_{y^j y^k}\phi+(\alpha_{y^j}s_{y^k}+\alpha_{y^k}s_{y^j})\phi_2+\alpha\phi_{22}s_{y^k}s_{y^j}+\alpha\phi_2 s_{y^j y^k}, \qquad (4.3)\\
F_{y^j y^k y^l} &= [(\alpha_{y^j y^k}s_{y^l}+\alpha_{y^j}s_{y^k y^l})\phi_2+(\alpha_{y^j}s_{y^l}+\alpha s_{y^j y^l})s_{y^k}\phi_{22}](j\to k\to l\to j)\\
&\quad+\alpha_{y^j y^k y^l}\phi+\alpha\phi_{222}s_{y^l}s_{y^k}s_{y^j}+\alpha\phi_2 s_{y^j y^k y^l}, \qquad (4.4)
\end{aligned}
$$

Plugging (3.6) and (3.12)-(3.14) into (4.3) and (4.4) yields

$$
\begin{aligned}
F_{y^j y^k} &= \frac{1}{\alpha}(\phi-s\phi_2)a_{jk}-\frac{s\phi_{22}}{\alpha^2}(b_k y_j+b_j y_k)+\frac{\phi_{22}}{\alpha}b_j b_k-\frac{1}{\alpha^3}(\phi-s\phi_2-s^2\phi_{22})y_j y_k, \qquad (4.5)\\
F_{y^j y^k y^l} &= \frac{1}{\alpha^2}\big[\frac{1}{\alpha}(s\phi_2+s^2\phi_{22}-\phi)a_{kl}y_j+\frac{s}{\alpha^2}(3\phi_{22}+s\phi_{222})b_k y_l y_j-s\phi_{22}a_{jl}b_k\\
&\quad-\frac{1}{\alpha}(\phi_{22}+s\phi_{222})b_l b_j y_k\big](j\to k\to l\to j)\\
&\quad+\frac{1}{\alpha^5}(\phi-3s\phi_2-6s^2\phi_{22}-s^3\phi_{222})y_j y_k y_l+\frac{1}{\alpha^2}\phi_{222}b_l b_k b_j. \qquad (4.6)
\end{aligned}
$$

Suppose F be of isotropic Berwald curvature, by (1.1), (4.5) and (4.6), we obtain

$$\begin{aligned}
B_j{}^i{}_{kl} = &\frac{\tau(x)}{\alpha}\big[(\phi - s\phi_2)a_{jk} - \frac{s\phi_{22}}{\alpha}(b_k y_j + b_j y_k) + \phi_{22}b_j b_k - \frac{1}{\alpha^2}(\phi - s\phi_2 - s^2\phi_{22})y_j y_k\big]\delta^i{}_l(j \to k \to l \to j)\\
&+\frac{\tau(x)}{\alpha^2}\big[\frac{1}{\alpha}(s\phi_2 + s^2\phi_{22} - \phi)a_{kl}y_j + \frac{s}{\alpha^2}(3\phi_{22} + s\phi_{222})b_k y_l y_j - s\phi_{22}a_{jl}b_k\\
&-\frac{1}{\alpha}(\phi_{22} + s\phi_{222})b_l b_j y_k\big]y^i(j \to k \to l \to j)\\
&+\frac{\tau(x)}{\alpha^2}\big[\frac{1}{\alpha^3}(3\phi - 3s\phi_2 - 6s^2\phi_{22} - s^3\phi_{222})y_j y_k y_l + \phi_{222}b_l b_k b_j\big]y^i. \qquad (4.7)
\end{aligned}$$

By (3.1) and (4.7), we obtain

$$T_1 + \alpha T_2 = 0, \qquad (4.8)$$

where

$$\begin{aligned}
T_1 := &\ \alpha^4\big\{[E - sE_2 - \rho(\phi - s\phi_2)]a_{kl}\delta^i_j + (E_{22} - \rho\phi_{22})b_l b_k\delta^i_j + (H_2 - sH_{22})a_{kl}b_j\big\}(j \to k \to l \to j)\\
&-\alpha^2[E - sE_2 - s^2E_{22} - \rho(\phi - s\phi_2 - s^2\phi_{22})](y_l\delta^i_j + a_{lj}y^i)y_k(j \to k \to l \to j) + \alpha^4 H_{222}b_l b_k b_j b^i\\
&-\alpha^2\big\{[E_{22} + sE_{222} - \rho(\phi_{22} + s\phi_{222})]b_k y^i + (H_2 - sH_{22} - s^2H_{222})y_k b^i)\big\}b_l y_j(j \to k \to l \to j)\\
&+[3E - 3sE_2 - 6s^2E_{22} - s^3E_{222} - \rho(3\phi - 3s\phi_2 - 6s^2\phi_{22} - s^3\phi_{222})]y_j y_k y_l y^i, \qquad (4.9)\\
T_2 := &-\alpha^2 s\big\{(E_{22} - \rho\phi_{22})[(y_k b_l + y_l b_k)\delta^i{}_j + a_{jl}b_k y^i] + [(H_2 - sH_{22})y_j a_{kl} + H_{222}b_k b_l y_j]b^i\big\}(j \to k \to l \to j)\\
&+s\big[3E_{22} + sE_{222} - \rho(3\phi_{22} + s\phi_{222})\big]y_l y_j b_k y^i(j \to k \to l \to j)\\
&+\alpha^2(E_{222} - \rho\phi_{222})b_l b_k b_j y^i + s(3H_2 - 3sH_{22} - s^2H_{222})y_j y_k y_l b^i. \qquad (4.10)
\end{aligned}$$

By (4.8), we know that

$$T_1 = 0, \quad T_2 = 0.$$

For $s \neq 0$, it follows from $T_2 y^j y^k y^l = 0$ that

$$(E_{222} - \rho\phi_{222})y^i - \alpha H_{222}b^i = 0. \qquad (4.11)$$

Both rational part and irrational part of (4.11) equal zero, namely

$$E_{222} - \rho\phi_{222} = 0, \quad H_{222} = 0. \qquad (4.12)$$

Plugging (4.12) into $T_2 = 0$ yields

$$\begin{aligned}
&-\alpha^2\big\{(E_{22} - \rho\phi_{22})[(y_k b_l + y_l b_k)\delta^i{}_j + a_{jl}b_k y^i] + (H_2 - sH_{22})a_{kl}y_j b^i\big\}(j \to k \to l \to j)\\
&+3(E_{22} - \rho\phi_{22})y_l y_j b_k y^i(j \to k \to l \to j) + 3(H_2 - sH_{22})y_j y_k y_l b^i = 0. \qquad (4.13)
\end{aligned}$$

Contracting (4.13) by $b^j b^k b^l$ yields

$$(E_{22} - \rho\phi_{22})b^2(b^2 - 3s^2)y^i + \alpha s[2b^2(E_{22} - \rho\phi_{22}) + (H_2 - sH_{22})(b^2 - s^2)]b^i = 0. \qquad (4.14)$$

Hence, it follows from (4.14) that

$$E_{22} - \rho\phi_{22} = 0, \quad H_2 - sH_{22} = 0. \qquad (4.15)$$

Inserting (4.12) and (4.15) into $T_1 = 0$ yields

$$[E - sE_2 - \rho(\phi - s\phi_2)]\big\{\alpha^2[\alpha^2 a_{kl}\delta^i_j - (y_l\delta^i_j + a_{lj}y^i)y_k](j \to k \to l \to j) + 3y_j y_k y_l y^i\big\} = 0. \qquad (4.16)$$

Multiplying (4.16) by $b^j b^k b^l$ yields

$$[E - sE_2 - \rho(\phi - s\phi_2)](b^2 - s^2)(\alpha b^i - sy^i) = 0. \qquad (4.17)$$

Hence, it is easy to see from (4.17) that

$$E - sE_2 - \rho(\phi - s\phi_2) = 0. \qquad (4.18)$$

Note that

$$E_{222}-\rho\phi_{222} = (E_{22}-\rho\phi_{22})_2, \quad s(E_{22}-\rho\phi_{22}) = -\big[E-sE_2-\rho(\phi-s\phi_2)\big]_2, \quad sH_{222} = -(H_2-sH_{22})_2.$$

Therefore, (4.18) implies that the first equalities of (4.12) and (4.15) hold. The second equality of (4.15) implies that the second equality of (4.12) holds. Moreover, if a general (α,β)-metric $F=\alpha\phi(b^2,s)$ is of isotropic Berwald curvature, then (4.1) and (4.2) hold.

Conversely, suppose that (4.1) and (4.2) hold. Setting $\psi := E-\rho(x)\phi$, then (4.1) is equivalent to

$$\psi - s\psi_2 = 0. \tag{4.19}$$

By solving Eq. (4.19), we obtain $\psi = \frac{1}{2}\sigma(b^2)s$. Hence,

$$E-\rho(x)\phi = \frac{1}{2}\sigma(b^2)s. \tag{4.20}$$

By (4.2), there exist two functions $t_1(b^2)$ and $t_2(b^2)$ such that

$$H = \frac{1}{2}\big[t_1(b^2)+t_2(b^2)s^2\big]. \tag{4.21}$$

By (3.5), (4.20) and (4.21),

$$\begin{aligned} G^i - \tau(x)Fy^i &= {}^{\alpha}G^i + c(x)\alpha E y^i + c(x)\alpha^2 H b^i - \tau(x)Fy^i \\ &= {}^{\alpha}G^i + \alpha\big[c(x)E-\tau(x)\phi\big]y^i + \frac{1}{2}c(x)\alpha^2\big[t_1(b^2)+t_2(b^2)s^2\big]b^i \\ &= {}^{\alpha}G^i + \frac{1}{2}c(x)\big\{\sigma(b^2)\beta y^i + [t_1(b^2)\alpha^2+t_2(b^2)\beta^2]b^i\big\}. \end{aligned} \tag{4.22}$$

Hence, G^i are quadratic in $y = y^i\frac{\partial}{\partial x^i}|_x$. On the other hand, it follows from (2.4) that

$$(G^i-\tau(x)Fy^i)_{y^jy^ky^l} = B_j{}^i{}_{kl} - \tau(x)(F_{y^jy^k}\delta^i{}_l + F_{y^jy^l}\delta^i{}_k + F_{y^ly^k}\delta^i{}_j + F_{y^jy^ky^l}y^i). \tag{4.23}$$

Using (4.22) and (4.23), we obtain that $F=\alpha\phi(b^2,s)$ is of isotropic Berwald curvature.

Observe that F is a Berwald metric if and only if its Berwald curvature $B_j{}^i{}_{kl}=0$. By (1.1), we obtain that F is a Berwald metric if and only if $\tau(x)=0$, i.e., $\rho(x)=0$. Hence, by the above process of proof, we get that F is a Berwald metric if and only if $E-sE_2 = H_2-sH_{22} = 0$. ☐

Proof of Theorem 1.1. Suppose that β satisfies (1.3), By Lemma 4.1, a general (α,β)-metric $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ is of isotropic Berwald curvature if and only if (4.1) and (4.2) hold. Then there exist three functions $\sigma(b^2)$, $t_1(b^2)$ and $t_2(b^2)$ such that (4.20) and (4.21) hold. Plugging (4.20)-(4.21) into (3.2) and (3.3) yields

$$\frac{\phi_2+2s\phi_1}{2\phi} - \frac{1}{2}(t_1+t_2s^2)\frac{s\phi+(b^2-s^2)\phi_2}{\phi} = \rho\phi+\frac{1}{2}\sigma s, \tag{4.24}$$

$$\frac{\phi_{22}-2(\phi_1-s\phi_{12})}{2\big[\phi-s\phi_2+(b^2-s^2)\phi_{22}\big]} = \frac{1}{2}\big(t_1+t_2s^2\big), \tag{4.25}$$

where we use ρ, σ, t_1 and t_2 instead of $\rho(x)$, $\sigma(b^2)$, $t_1(b^2)$ and $t_2(b^2)$, respectively. (4.24) and (4.25) are equivalent to

$$\big[1-(t_1+t_2s^2)(b^2-s^2)\big]\phi_2 + 2s\phi_1 - s\big[(t_1+t_2s^2)+\sigma\big]\phi - 2\rho\phi^2 = 0, \tag{4.26}$$

$$\big[1-(b^2-s^2)(t_1+t_2s^2)\big]\phi_{22} - 2\phi_1 + 2s\phi_{12} + s(t_1+t_2s^2)\phi_2 - (t_1+t_2s^2)\phi = 0. \tag{4.27}$$

Differentiating (4.26) with respect to s yields

$$\begin{aligned} &\big[1-(t_1+t_2s^2)(b^2-s^2)\big]\phi_{22} + 2\phi_1 + 2s\phi_{12} + s\big(t_1-\sigma-2b^2t_2+3t_2s^2\big)\phi_2 \\ &-\big(t_1+\sigma+3t_2s^2\big)\phi - 4\rho\phi\phi_2 = 0. \end{aligned} \tag{4.28}$$

From (4.28) $-$ (4.27), we obtain

$$4\phi_1 - s\big[2t_2(b^2 - s^2) + \sigma\big]\phi_2 - (\sigma + 2t_2 s^2)\phi - 4\rho\phi\phi_2 = 0. \tag{4.29}$$

From (4.26) $\times$ 2 $-$ (4.29) $\times$ s, we obtain

$$\big[2 - 2t_1(b^2 - s^2) + \sigma s^2\big]\phi_2 - (2t_1 + \sigma)s\phi - 4\rho\phi^2 + 4\rho s\phi\phi_2 = 0. \tag{4.30}$$

Note that (4.30) is equivalent to

$$\Big(\frac{2 - 2t_1(b^2 - s^2) + \sigma s^2}{\phi^2}\Big)_2 + \Big(\frac{8\rho s}{\phi}\Big)_2 = 0. \tag{4.31}$$

Case 1 $\rho \neq 0$

1) $2 - 2t_1(b^2 - s^2) + \sigma s^2 \neq 0$

Integrating (4.31) yields

$$\phi = \frac{4\rho s + \sqrt{2k(1 - t_1 b^2) + (16\rho^2 + 2kt_1 + k\sigma)s^2}}{k}, \tag{4.32}$$

where $k = k(b^2)$ is any non-zero smooth function. Then the corresponding general (α, β)-metric $F = \alpha\phi(b^2, s)$ is a Randers metric.

2) $2 - 2t_1(b^2 - s^2) + \sigma s^2 = 0$

In this case, (4.31) is reduced to $(\frac{s}{\phi})_2 = 0$, it is easy to obtain that $\phi = \frac{s}{a(b^2)}$. Hence, the corresponding general (α, β)-metric $F = \alpha\phi(b^2, s)$ is a Kropina metrics.

Case 2 $\rho = 0$

In this case, it follows from (1.1) that F is a Berwald metric. (4.30) is reduced to

$$\big[2 - 2t_1(b^2 - s^2) + \sigma s^2\big]\phi_2 - (2t_1 + \sigma)s\phi = 0. \tag{4.33}$$

i) $2 - 2t_1(b^2 - s^2) + \sigma s^2 \neq 0$

By (4.33), we obtain

$$\phi = t_3(b^2)\sqrt{2(1 - b^2 t_1) + (\sigma + 2t_1)s^2}, \tag{4.34}$$

where $t_3(b^2)$ is any positive smooth function. Hence, in this case, the corresponding general (α, β)-metric $F = \alpha\phi(b^2, s)$ is a Riemannian metric.

ii) $2 - 2t_1(b^2 - s^2) + \sigma s^2 = 0$

Note that $\phi > 0$ and $s \neq 0$. In this case, (4.33) is equivalent to

$$\sigma + 2t_1 = 0, \quad 2 - 2(b^2 - s^2)t_1 + \sigma s^2 = 0. \tag{4.35}$$

By (4.35), we have

$$\sigma = -\frac{2}{b^2}, \quad t_1 = \frac{1}{b^2}. \tag{4.36}$$

In this case, (4.29) imply that (4.26) holds. By the above caculations, it is easy to see that (4.26) and (4.29) imply (4.27). Therefore, we only need to solve (4.29). Plugging (4.36) into (4.29) yields

$$\phi_1 + \frac{1}{2}s\Big[\frac{1}{b^2} - (b^2 - s^2)t_2\Big]\phi_2 = \frac{1}{2}\Big(-\frac{1}{b^2} + t_2 s^2\Big)\phi. \tag{4.37}$$

The characteristic equation of PDE (4.37) is

$$\frac{db^2}{1} = \frac{ds}{\frac{1}{2}s[\frac{1}{b^2} - (b^2 - s^2)t_2]} = \frac{d\phi}{\frac{1}{2}(-\frac{1}{b^2} + t_2 s^2)\phi}. \tag{4.38}$$

Firstly, we solve

$$\frac{db^2}{1}=\frac{ds}{\frac{1}{2}s[\frac{1}{b^2}-(b^2-s^2)t_2]}. \tag{4.39}$$

(4.39) is equivalent to

$$\frac{ds}{db^2}=\frac{1}{2}(\frac{1}{b^2}-b^2t_2)s+\frac{1}{2}t_2s^3.$$

This is a Bernoulli equation which can be rewritten as

$$\frac{d}{db^2}\left(\frac{1}{s^2}\right)=\left(b^2t_2-\frac{1}{b^2}\right)\frac{1}{s^2}-t_2.$$

This is a linear 1-order ODE of $\frac{1}{s^2}$. One can easily get its solution

$$\frac{1}{s^2}=e^{\int(b^2t_2-\frac{1}{b^2})db^2}\left[\tilde{c}_1-\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2\right], \tag{4.40}$$

where $\tilde{c}_1$ is a constant. Hence, by (4.40), one independent integral of Eq. (4.38) is

$$\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2}=\frac{1}{\tilde{c}_1}. \tag{4.41}$$

Note that the characteristic equation (4.38) is equivalent to

$$\frac{db^2}{1}=\frac{d\ln s}{\frac{1}{2}[\frac{1}{b^2}-(b^2-s^2)t_2]}=\frac{d\ln\phi}{\frac{1}{2}(-\frac{1}{b^2}+t_2s^2)}. \tag{4.42}$$

Eq. (4.42) implies

$$\frac{db^2}{1}=\frac{d\ln s-d\ln\phi}{\frac{1}{b^2}-\frac{1}{2}b^2t_2} \tag{4.43}$$

By integrating Eq. (4.43), we obtain another independent integral of Eq. (4.38)

$$\ln\frac{s}{\phi}-\int(\frac{1}{b^2}-\frac{1}{2}b^2t_2)db^2=\tilde{c}_2, \tag{4.44}$$

where $\tilde{c}_2$ is a constant. Hence, the general solution of Eq. (4.37) is

$$\Phi\left(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2},\ln\frac{s}{\phi}-\int(\frac{1}{b^2}-\frac{1}{2}b^2t_2)db^2\right)=0, \tag{4.45}$$

where $\Phi(\xi,\eta)$ is any continuously differentiable function. Suppose $\Phi'_\eta\neq 0$, then we can solve from (4.45) that

$$\phi=\varphi(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2})e^{\int(\frac{1}{2}b^2t_2-\frac{1}{b^2})db^2}s, \tag{4.46}$$

where $\varphi(\cdot)$ is any positive continuously differentiable function. Hence, the corresponding general (α,β)-metric is

$$F=\alpha\varphi(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2})e^{\int(\frac{1}{2}b^2t_2-\frac{1}{b^2})db^2}s.$$

$\square$

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